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Geometric Quantization in the Presence of an Electromagnetic Field

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Some aspects of the formalism of geometric quantization are described emphasizing the role played by the symmetry group of the quantum system which, for the free particle, turns out to be a central extension $\tilde{G}_{(m)}$ of the Galilei group G. The resulting formalism is then applied to the case of a particle interacting with the electromagnetic field, which appears as a necessary modification of the connection 1-form of the quantum bundle when its invariance group is generalized to a *local* extension of G. Finally, the quantization of the electric charge in the presence of a Dirac monopole is also briefly considered.

1. INTRODUCTION

Differential geometry is a branch of mathematics which has always exerted a profound and clarifying influence in the formulation of basic theories of physics. Aside from the classical example of Einstein's general theory of relativity, it has recently been shown of great importance in other areas of physics (see, e.g., Hermann, 1973) and particularly for the precise understanding of certain aspects of gauge theories (Yang, 1975, 1977; Mayer and Trautman, 1981). Another relatively recent application is the formalism of geometric quantization of Souriau (1970) and Kostant (1970) (see also Simms and Woodhouse, 1976; García-Peréz, 1978; Abraham and Marsden, 1978; and Śniatycki, 1980). Such a formalism is at present of no experimental significance but it constitutes an interesting attempt towards establishing a strict relationship between the classical theory, precisely formulated in a geometrical manner, and the quantum theory, usually defined in a more vague mathematical form further away from physical intuition and only connected with its parent classical theory through not sufficiently precise statements of correspondence.

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This paper has the purpose of exhibiting some simple but interesting implications of the geometric quantization formalism and, in particular, of analyzing the role of the symmetry group of the quantum system. With this aim, a *quantum manifold* (P, Θ) is defined (Section 3) as a principal bundle $P(S, U(1))$ endowed with a connection form Θ such that it induces in the base manifold S a symplectic structure (S, ω). The condition for a symplectic manifold (S, ω) to be *quantizable* is that the 2-form ω be of integer class, i.e., $[\omega] \in H^2(M, Z)$ (Section 3); this integrality condition is used in Section 8 to recover the well-known Dirac quantization of the electric charge in the presence of a magnetic monopole.

The trivial case of the free particle is discussed in Section 4 as a first example of quantization of a system; there, the different manifolds previously mentioned are constructed. In the process, the question of extending the symmetry group to obtain the corresponding one of the quantum system will naturally appear; Section 5 is devoted to explicitly realizing the extended Galilei group which acts on the quantum manifold P and of which it is its symmetry group, as indicated by the fact that the connection form is left strictly invariant under its action. The generalization of the global extended Galilei invariance in the $U(1)$ subgroup will lead to a necessary modification of the connection form, which will accommodate in this way the electromagnetic field $(\phi, \mathbf{A})$. This will be described in Section 6; as expected, the modification of the connection form is the one dictated by the minimal substitution prescription, which in this way will appear as a consequence of the geometric quantization formalism. The quantum particle in the presence of a magnetic field is considered in Section 7. Finally, Section 8 will consider the case of magnetism with monopoles, which is briefly described by making use of Yang's approach for the potentials and incorporates, as mentioned above, the geometric quantization of the electric charge.

2. THE FORMALISM OF GEOMETRIC QUANTIZATION

We start with a short recapitulation of the formalism of geometric quantization (Souriau, 1970; Kostant, 1970; Simms and Woodhouse, 1976; García-Peréz, 1978; Abraham and Marsden, 1978; Śniatycki, 1980) which we shall present in a slightly different form which is more suitable for our purposes. In fact, in this paper we shall restrict ourselves to considerations on the quantum manifold, disregarding other ingredients of the geometric quantization process such as polarizations, quantum operators or metaplectic structures.

Let P be a differentiable manifold of odd dimension $2n+1$ on which the group $U(1)$ (the torus S^1) acts effectively. Let us call $\xi[z]$ the elements of $P[U(1)]$, Z_P the action of $z \in U(1)$ on P, and Ξ_P the vector field of $U(1)$

which generates its action on P. Then P is said to constitute a *quantum manifold* (Souriau, 1970) if there is a $U(1)$-invariant 1-form Θ on P such that (P,Θ) is a contact manifold on P. Thus (P,Θ) is a quantum manifold iff

(1) $Z_P\xi=\xi, \xi\in P \Leftrightarrow z=\mathbb{1}\in U(1)$ [effective action of $U(1)$]

(2) $\dim(\ker\Theta\cap\ker d\Theta)=0, \quad \dim(\ker d\Theta)=1$

(contact structure: Θ is of constant class $2n+1$)

and *the* dynamical vector field associated with the contact form Θ is Ξ_P, i.e.,

(2′) $i_{\Xi_P}\Theta=1, \quad i_{\Xi_P}d\Theta=0$

The above condition (2′) implies $L_{\Xi_P}\Theta=0$ and $L_{\Xi_P}d\Theta=0$. It is thus clear that the presympletic form $d\Theta$ on P defines on the quotient manifold P/Ξ_P a sympletic form ω so that $(P/\Xi_P,\omega)$ is a *symplectic manifold.*

The above properties of the contact 1-form Θ and P may be summarized in the following proposition:

Proposition. A quantum manifold is a principal bundle $P(S,U(1))$ on which a connection form Θ has been defined such that the associated curvature form $\Omega=D\Theta$ defines on the base manifold $S\approx P/U(1)$ a sympletic structure (S,ω).

Proof. To prove the above assertion, let us recall (Kobayashi and Nomizu, 1963) that a connection form γ on a principal bundle $P(S,G)$ is a 1-form on P valued on the vector space $\mathcal{G}$ (the Lie algebra of the structure group G), $\gamma:\tau(P)\to\mathcal{G}$ such that

(a) $\gamma(Z_P)=Z$ ($Z_P\in\tau(P)$, the tangent bundle to P; $Z\in\mathcal{G}$)

(b) $L_z^*\gamma=\mathrm{ad}(z^{-1})\gamma, \forall z\in G$

and that the curvature 2-form associated with γ is given by

$$\Omega=D\gamma=d\gamma+[\gamma,\gamma]$$

In our case, $G\equiv U(1), \mathcal{G}=\mathbb{R}$ and putting $\Theta=\gamma$ one immediately finds that conditions (a) and (b) above read

(a) $\Theta(\Xi_P)=1$, (i.e., $i_{\Xi_P}\Theta=1$)

(b) $L_z^*\Theta=\Theta$, i.e., $L_{\Xi_P}\Theta=0\Rightarrow i_{\Xi_P}d\Theta=0$

This shows that condition (2′) above is satisfied. Condition (1) is trivially fulfilled since P is a principal bundle. Finally, if the curvature 2-form $\Omega=d\Theta+[\Theta,\Theta]=d\Theta$ has to define a symplectic form $d\Theta/U(1)\equiv\omega$ on the base manifold it is sufficient to require that Ξ_P be the *only* vector field which satisfies $i_{\Xi_P}d\Theta=0$, i.e., that (P,Θ) be a contact manifold. Thus, (2) is also satisfied. ■

The above definition of a quantum manifold as a principal bundle with connection form Θ now allows us to define a quantomorphism [i.e., a isomomorphism of quantum manifolds, see Souriau (1970)] as an isomorphism between principal bundles $P(S,U(1))$, $P'(S',U(1))$ compatible with the connections Θ,Θ'. Thus a diffeomorphism $F: P\to P'$ is a *quantomor-*

phism iff

$$F^*\Theta = \Theta'$$

$$F(z_P \xi) = z_{P'} F(\xi) \qquad \forall \xi \in P, \qquad z \in U(1)$$

where $Z_{P'}$ is the image of Z_P by the group isomorphism induced by F. Clearly, F also induces a base diffeomorphism because of the commutativity of the diagram

$$\begin{array}{ccc} P & \xrightarrow{F} & P' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ S & \xrightarrow[F_b]{} & S \end{array}$$

which is a symplectomorphism. If $F_b = I$ (F is a quantomorphism over S) S and S' will be identified, and then the quantizations are *equivalent quantizations* of the symplectic manifold (S, ω).

3. EXISTENCE AND CLASSIFICATION OF QUANTIZATIONS

The existence of quantum manifolds (P, Θ) on a symplectic manifold (S, ω) is determined (Kostant, 1970) by the following theorem

> *Theorem.* Let (S, ω) be a symplectic manifold. The necessary and sufficient condition for a quantization (P, Θ) of (S, ω) to exist, is that $[\omega] \in \operatorname{Im} \varepsilon$, where $[\omega]$ is the de Rham cohomology class and ε is the canonical map $\varepsilon: H^2(S, \mathbb{Z}) \to H^2(S, \mathbb{R})$ induced by the inclusion $\varepsilon: \mathbb{Z} \to \mathbb{R}$. This condition is usually expressed by saying, for short, that ω is of integer class.

Proof.

Direct theorem. Let ω be a symplectic form on S of integer class and let $\{U_\alpha\}$ be an open cover of S. Because ω is closed, there exist 1-forms Λ_α on the open sets U_α such that

$$\omega|_{U_\alpha} = d\Lambda_\alpha \qquad \omega|_{U_\beta} = d\Lambda_\beta \tag{1}$$

On $U_\alpha \cap U_\beta \neq \phi$,

$$\omega|_{U_\alpha} = \omega|_{U_\beta} \Rightarrow d(\Lambda_\alpha - \Lambda_\beta) = 0 \tag{2}$$

and so there exists on $U_\alpha \cap U_\beta$ a function $f_{\alpha\beta}$ such that $\Lambda_\alpha - \Lambda_\beta = df_{\alpha\beta}$; obviously, $df_{\alpha\beta} = -df_{\beta\alpha}$. On $U_\alpha \cap U_\beta \cap U_\gamma \neq \phi$ we have

$$df_{\alpha\beta} - df_{\alpha\gamma} + df_{\beta\gamma} = 0 \tag{3}$$

so that

$$f_{\alpha\beta} - f_{\alpha\gamma} - f_{\beta\gamma} \equiv C_{\alpha\beta\gamma} \in \mathbb{R} \tag{4}$$

is a 2-cocycle (indeed, it is the differential of the 1-cochain $f_{\alpha\beta}$ and, on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\sigma \neq \phi$, $C_{\alpha\beta\gamma}$ satisfies $C_{\beta\gamma\sigma} - C_{\alpha\gamma\sigma} + C_{\alpha\beta\sigma} - C_{\alpha\beta\gamma} = 0$).

Because ω is of integer class, $C_{\alpha\beta\gamma} \in \mathbb{Z}$ (In fact, ω leads to a number of elements of $H^2(S, \mathbb{Z})$ characterized by $\operatorname{Ker} \varepsilon$). From (4) one obtains

$$f_{\alpha\beta} - f_{\alpha\gamma} + f_{\beta\gamma} \in \mathbb{Z} \tag{5}$$

Now, we define an element $z_{\alpha\beta}$ of $H^1(\{U_\alpha\}, U(1))$ by

$$z_{\alpha\beta} \equiv \exp(2\pi i f_{\alpha\beta}) \tag{6}$$

($z_{\alpha\beta}$ is obviously a 1-cocycle; indeed, because of (5) $(\partial z)_{\alpha\beta\gamma} = z_{\alpha\beta} \cdot z_{\alpha\gamma}^{-1} \cdot z_{\beta\gamma} = 1 \in U(1)$). Let $P(S, U(1))$ be a principal bundle on S determined by the transition functions $z_{\alpha\beta}$. The family of 1-forms Λ_α on S satisfy

$$\Lambda_\beta - \Lambda_\alpha = df_{\alpha\beta} = \frac{dz_{\alpha\beta}}{2\pi i z_{\alpha\beta}} \tag{7}$$

and, consequently, define a connection form Θ on $P(S, U(1))$ so that $(P, U(1))$ is a quantum manifold. [In particular, if $z_{\alpha\beta}$ is a coboundary, it may be written as $z_{\alpha\beta} = z_\alpha \cdot z_\beta^{-1}$ and, accordingly,

$$\frac{dz_{\alpha\beta}}{2\pi i z_{\alpha\beta}} = \frac{dz_\alpha}{2\pi i z_\alpha} - \frac{dz_\beta}{2\pi i z_\beta} \tag{8}$$

i.e.,

$$\Lambda_\alpha + \frac{dz_\alpha}{2\pi i z_\alpha} = \Lambda_\beta + \frac{dz_\beta}{2\pi i z_\beta}. \tag{9}$$

Thus, if $\{h_\alpha\}$ is a partition of unity subordinated to $\{U_\alpha\}$ the connection 1-form is explicitly given by

$$\Theta = \sum_\alpha h_\alpha \left(\Lambda_\alpha + \frac{dz_\alpha}{2\pi i z_\alpha} \right) \tag{10}$$

Reciprocal theorem. Let (P, Θ) be a quantum manifold in the sense of ∮1. Because Θ is a connection form on P, it defines a family of 1-forms on the base manifold S, $\Lambda_\alpha = \sigma_\alpha^* \Theta$, which satisfy

$$\Lambda_\beta = ad\left(z_{\alpha\beta}^{-1}\right)\Lambda_\alpha + \theta_{\beta\alpha} \tag{11}$$

(see, e.g. Kobayashi and Nomizu, 1963, p. 66 for the meaning of σ_α and $\theta_{\beta\alpha}$). In our case $G = U(1)$ and (11) simply reads

$$\Lambda_\beta = \Lambda_\alpha + \frac{dz_{\alpha\beta}}{2\pi i z_{\alpha\beta}} \tag{12}$$

Thus, identifying $dz_{\alpha\beta}/2\pi i z_{\alpha\beta}$ with $df_{\alpha\beta}$ the condition of 1-cocycle for $z_{\alpha\beta}$ imposes on the 2-cocycle $C_{\alpha\beta\gamma}$ given by (4) the condition of being of integer class and to the symplectic form $\omega \equiv d\Theta/U(1)$ the associated integrality condition. ■

The following proposition now holds:

Proposition. The set of inequivalent quantizations of a symplectic manifold (S, ω) is parametrized by the group $H^1(S, U(1))$.

Proof.

As it was indicated in the above theorem, the class of ω, $[\omega] \in H^2(S, \mathbb{R})$, determines a set in $H^2(S, \mathbb{Z})$ in one to one correspondence with the group $\ker \varepsilon$. The elements of this group characterize the possible nonequivalent principal bundles $H^2(S, \mathbb{Z})$ on which a connection Θ (such that $d\Theta/U(1) = \omega$) may be defined. Note that in terms of associated bundles, $\ker \varepsilon$ parametrizes the possible Chern classes in $H^2(S, \mathbb{Z})(\approx \check{H}^1(S, U(1)))$ which admit a Θ with the above condition

Now, once a given Chern class has been fixed (and thus a given class $[P]$ of equivalent principal bundles $P(S, U(1))$) one may ask how many Θ may be defined for the same ω. It is then realized that to a given connection Θ on $P \in [P]$ it is possible to add any closed (flat) 1-form γ on S; Θ and $\Theta' = \Theta + \gamma$ have the same curvature ω. Nevertheless, Θ and Θ' are not always inequivalent connections: $\Theta \sim \Theta' \Rightarrow \Theta - \Theta' = d\phi/2\pi i \phi$ [cfr. (7)]. Such a condition is fulfilled when $[\gamma] \in H^1(S, \mathbb{R})$ is of integer class. Indeed in this case we have

$$\gamma|_{U_\alpha} = df_\alpha \qquad \gamma|_{U_\beta} = df_\beta \qquad f_\beta - f_\alpha \in \mathbb{Z}. \tag{13}$$

Thus, the functions $\phi_\alpha = \exp(2\pi i f_\alpha)$ define in fact a global function ϕ on S (they coincide on $U_\alpha \cap U_\beta$) and thus $\gamma = d\phi/2\pi i \phi$. Since in this case $\varepsilon: H^1(S, \mathbb{Z}) \to H^1(S, \mathbb{R})$ is injective, we find that given the class $[P]$ there are

$H^1(S,\mathbb{R})/H^1(S,\mathbb{Z})$ inequivalent quantizations (note that if γ is exact, Θ and Θ' are trivially equivalent). Thus, the result of the proposition may be demonstrated (García-Pérez, 1978) by the following series of sequences:

$$0 \to \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{R} \xrightarrow{\exp(2\pi i f)} U(1) \to 1$$

induces

$$H^1(S,\mathbb{R}) \xrightarrow{\exp} H^1(S,U(1)) \xrightarrow{\partial \circ \log} H^2(S,\mathbb{Z}) \xrightarrow{\varepsilon} H^2(S,\mathbb{R})$$

and this is made an exact cohomology sequence by writing

$$0 \to H^1(S,\mathbb{R})/H^1(S,\mathbb{Z}) \xrightarrow{\exp} H^1(S,U(1)) \xrightarrow{\partial \circ \log} \ker \varepsilon \to 0 \qquad \blacksquare$$

Now, it is known that $H^1(S,U(1)) \approx \mathrm{Hom}(\pi_1(S),U(1))$; thus, there exist as many inequivalent quantizations of (S,ω) as characters they are for the first homotopy group of the manifold S, $\pi_1(S)$. From this result follows immediately that a S admits a unique quantization iff S is simply connected (Souriau, 1970; Kostant, 1970).

We shall not consider here the question of the definition of the Hilbert space associated with the state of the quantum system, for which refer e.g., to (Simms and Woodhouse, 1976; García-Pérez, 1978; Abraham and Marsden, 1978; Śniatycki, 1980), and move on directly to the applications of the formalism above developed.

4. THE CASE OF THE FREE GALILEAN PARTICLE

We start the process by extending the evolution manifold $E = T^*(M) \times \mathbb{R}$ by $S^1[\approx U(1)]$ by direct product, $W = E \times S^1$ (Souriau, 1970). Then, we shall extend the Poincaré–Cartan Θ_H 1-form on E to W, and obtain P by taking the quotient of W by the characteristic field X_{Θ_W} of Θ_W. Schematically,[1]

$$(E = T^*(M) \times R, \Theta_H) \xrightarrow{\text{ext by } U(1)} (W, \Theta_W) \xrightarrow[X_{\Theta_W}]{\text{quotient by}} (P, \Theta_P)$$

[1]We might follow alternatively the (perhaps) more natural steps

$$E \xrightarrow[\tilde{X}_H]{\text{quot. by}} U_H \xrightarrow{\text{ext. by } S^1} P$$

where $\tilde{X}_H$ is the Hamiltonian field and U_H the solution manifold of $\tilde{X}_H$.

With the *global* parametrization for $W\{t, x^i, p_j, \zeta\}$ $\zeta \in \mathbb{C}$, $|\zeta| = 1$ the contact 1-form on E is trivially extended to W by writing

$$\Theta_W = p_i\, dx^i - H\, dt + d\zeta / i\zeta$$

$$d\Theta_W = dp_i \wedge dx^i - dH \wedge dt \tag{14}$$

The characteristic field $X_{\Theta_W} \in \mathfrak{X}(W)$ is determined by

$$i_X \Theta_W = 0 = i_X d\Theta_W \tag{15}$$

Writing X generically as

$$X = X^i \frac{\partial}{\partial x^i} + X_{i*} \frac{\partial}{\partial p_i} + X_t \frac{\partial}{\partial t} + i\zeta X_\zeta \frac{\partial}{\partial \zeta} \tag{16}$$

equations (15) give

$$X^i = \frac{p^i}{m} X_t, \qquad X_{i*} = 0, \qquad X_\zeta = \frac{-\mathbf{p}^2}{2m} X_t, \qquad X_t = X_t \tag{17}$$

Putting $X_t = 1$, the above equations may be integrated with the result

$$\frac{dx^i}{d\lambda} = \frac{p^i}{m}, \qquad \frac{1}{i\zeta} \frac{d\zeta}{d\lambda} = -\frac{\mathbf{p}^2}{2m}$$

$$\frac{dp_i}{d\lambda} = 0, \qquad \frac{dt}{d\lambda} = 1$$

$$x^i = \frac{p^i}{m} t + C^i, \qquad \zeta = c \exp\left(-i \frac{\mathbf{p}^2}{2m} t \right)$$

$$p_i = C_{i*}, \qquad t = \lambda \tag{18}$$

Relabeling K^i, P_i, z the integration constants C^i, C_{i*}, c the manifold of solutions P is now parametrized by

$$K^i = \frac{p^i}{m} t - x^i, \qquad P_i \equiv p_i, \qquad z = \zeta \exp\left(i \frac{\mathbf{p}^2}{2m} t \right) \tag{19}$$

Now, the 1-form Θ_W on W defines on the quotient manifold $P \equiv W/\Theta_W$ the 1-form

$$\Theta = P_i dK^i + \frac{dz}{iz} \tag{20}$$

Θ defines a quantization (P, Θ) of U_H; clearly, $d\Theta/U(1)$ is given by

$$d\Theta/U(1) = \omega = dP_i \wedge dK^i = d\left(P_i dK^i\right) \tag{21}$$

which is clearly of integer class (in fact, of the class zero).

5. EXTENDED GALILEAN INVARIANCE OF THE FREE QUANTUM PARTICLE

The geometric quantization formalism previously introduced has not yet made any reference to the invariance group of the physical system. It is clear, however, that the transition from the classical manifold S to the quantum one P should be made in a way giving rise to compatible actions of the symmetry group on both S and P. But, already for the most simple quantum system, the free particle, the relation between the quantization procedure and the symmetry group appears to be deeper than a simple compatibility condition: the quantum symmetry group [the extended—by $U(1)$—Galilei group $\tilde{G}_{(m)}$] is obtained by a suitable—central—extension of the ordinary Galilei group G, which does *not* include G as a subgroup (see in this connection Aldaya and Azcárraga, 1981). This fact underlines the close relation between the nature of the symmetry group and the character—quantum or classical—of the dynamical system. Such a relation deserves a closer analysis, which will be developed elsewhere (Aldaya and Azcárraga, 1982).

In fact, and as already pointed out by Souriau (1970), the connection between the quantum group and the classical one is dictated by its cohomological structure. For the case of the Galilei group, the fact that $H^2(G, U(1)) \neq 0$ is the reason why the relation among the brackets of the Lie algebra of G and the associated dynamical Poisson brackets is not an algebra isomorphism. That is also the reason why the dynamical system associated with a classical free particle is not strictly invariant under G, but presents semi-invariance under the boosts; as is well known (Landau and Lifshitz, 1960; Lévy-Leblond, 1969), no classical Lagrangian can be found strictly invariant under the whole group G. The same can be stated of the Poincaré–Cartan form Θ associated with the classical system. Thus we are led in a natural way to require that in applying the geometric quantization formalism *both* the classical manifold and the (Galilei) symmetry group should be extended [by S^1 and $U(1)$] in order to obtain a group of trivial cohomology under which the strict invariance of the quantum system can be imposed as a postulate of the free quantum theory. In this sense, it could be asserted that the quantum theory is predetermined by the quantum symmetry group.[2]

Let us explicitly express the symmetries of the quantization 1-form Θ on $W \equiv (T^*(\mathbb{R}^3)\times\mathbb{R})\times S^1$, where $T^*(\mathbb{R}^3)\times\mathbb{R}$ is the extension of the phase space $T^*(\mathbb{R}^3)$ by the time. Using $(t,\mathbf{x},\mathbf{p},\zeta)$ as *global* coordinates for W ($|\zeta|=1$), the finite transformations of the extended Galilei group $\tilde{G}_{(m)}$ are given by

$$\begin{aligned} t' &= t+\tau \\ \mathbf{x}' &= R\mathbf{x}+\mathbf{a}+\mathbf{v}t \\ \mathbf{p}' &= R\mathbf{p}+m\mathbf{v} \\ \zeta' &= \zeta\exp\{-i[\tfrac{1}{2}mv^2t+(R\mathbf{x})\cdot m\mathbf{v}+\theta]\} \end{aligned} \tag{22}$$

where the last equation may be modified by adding to the exponent a coboundary on G [see in this respect Lévy-Leblond (1964), Bargmann (1954)] and where $(\tau,\mathbf{a},\mathbf{v},R,\theta,)$ are the eleven parameters which characterize in an obvious manner the transformations of $\tilde{G}_{(m)}$, θ corresponding to its central $U(1)$ subgroup. By differentiating the tranformation law (22) at the unity $(0,\mathbf{0},\mathbf{0},I,0)\in\tilde{G}_{(m)}$, the generators of the extended Galilei group are obtained:

$$\begin{aligned} P_{(0)} &= \frac{\partial}{\partial t} \\ P_{(i)} &= \frac{\partial}{\partial x^i} \\ J_{(i)} &= \varepsilon_{ij\cdot}^{\;\;k}x^j\frac{\partial}{\partial x^k}+\varepsilon_{ij\cdot}^{\;\;k}p^j\frac{\partial}{\partial p^k} \\ K_{(i)} &= t\frac{\partial}{\partial x^i}+m\frac{\partial}{\partial p^i}-imx_i\zeta\frac{\partial}{\partial\zeta} \\ X_{(\theta)} &= -i\zeta\frac{\partial}{\partial\zeta} \end{aligned} \tag{23}$$

[2]The Poincaré group, having $H^2(P,U(1))=0$, is less constraining in the definition of the quantum system.

The invariance of Θ_W [14] may now be checked by evaluating $L_X\Theta_W$, with the result that

$$L_X\Theta_W = 0 \qquad \forall X \in \tilde{\mathcal{G}}_{(m)} \tag{24}$$

where $\tilde{\mathcal{G}}_{(m)}$ is the Lie algebra of $\tilde{G}_{(m)}$.

Let us now repeat the process by checking the invariance of the quantum manifold $P \overset{\pi}{\to} S$. Using the coordinates (Section 4) $(\mathbf{K}, \mathbf{P}, z)$, $|z| = 1$, where $\mathbf{K}, \mathbf{P}$ parametrize the symplectic manifold of the classical solutions, the action of $\tilde{G}_{(m)}$ is given by

$$\begin{aligned}
\mathbf{K}' &= R\mathbf{K} + \mathbf{a} - \tau\left(R\frac{\mathbf{P}}{m} + \mathbf{v}\right) \\
\mathbf{P}' &= R\mathbf{P} + m\mathbf{v} \\
z' &= z\exp - i\left[(R\mathbf{K})\cdot m\mathbf{v} - \tau\left(\frac{\mathbf{P}^2}{2m} + R\mathbf{P}\cdot\mathbf{v} + \frac{1}{2}m\mathbf{v}^2\right) + \theta\right]
\end{aligned} \tag{25}$$

so that differentiating again at the unity of $\tilde{G}_{(m)}$ we obtain

$$\begin{aligned}
P_{(0)} &= -\frac{P^i}{m}\frac{\partial}{\partial K^i} + \frac{\mathbf{P}^2}{2m} iz\frac{\partial}{\partial z} \\
P_{(i)} &= \frac{\partial}{\partial K^i} \\
J_{(i)} &= \varepsilon_{ij\cdot}^{\ \ k} K^j \frac{\partial}{\partial K^k} + \varepsilon_{ij\cdot}^{\ \ k} P^j \frac{\partial}{\partial P^k} \\
K_{(i)} &= m\frac{\partial}{\partial P^i} - mK_i iz\frac{\partial}{\partial z} \\
X_{(\theta)} &= -iz\frac{\partial}{\partial z}
\end{aligned} \tag{26}$$

A simple calculation now shows that, on P,

$$L_X\Theta = 0 \qquad \forall X \in \tilde{\mathcal{G}}_{(m)} \tag{27}$$

where now $\Theta = P_i dK^i + dz/iz$. We conclude this section by noting the trivial structure of Θ as the connection 1-form on the principal—trivial—bundle $P \equiv S \times U(1)$ of structure group $U(1)$.

6. INTERACTION OF A GALILEAN PARTICLE WITH THE ELECTROMAGNETIC FIELD AND LOCAL EXTENSION OF THE GALILEI GROUP G

The group geometrical approach to quantization already described is based upon extending the Galilei symmetry group G to the 11-parameter group $\tilde{G}_{(m)}$ and adding simultaneously the variable $\zeta \in S^1$ to the classical manifold so that Θ turns out to be strictly invariant. The question of extending this invariance to a local invariance in the $U(1)$ (phase) subgroup naturally presents itself, together with the associated problem of finding a new connection form invariant under this new group $\tilde{G}_{(m)}(t, \mathbf{x})$, where the bracket indicates the local nature of the phase group which will be denoted by $U(1)(t, \mathbf{x})$.

The Lie algebra of $U(1)(t, x)$ is the tensor product of the Lie algebra of $U(1)$ and the algebra of differentiable functions $\mathcal{F}(\mathbb{R}^3 \times \mathbb{R})$ on $\mathbb{R}^3 \times \mathbb{R}$. If f is an arbitrary element of $\mathcal{F}(\mathbb{R}^3 \times \mathbb{R})$, the algebra of $\tilde{G}_{(m)}(t, x)$ is characterized by the commutation relations

$$\begin{gathered} [J_i, J_j] = \varepsilon_{ij}^{\cdot\cdot k} J_k \qquad [J_i, K_j] = \varepsilon_{ij}^{\cdot\cdot k} K_k \qquad [J_i, P_j] = \varepsilon_{ij}^{\cdot\cdot k} P_k \\ [J_i, H] = [K_i, K_j] = [P_i, P_j] = [P_i, H] = 0 \\ [K_i, H] = P_i, \qquad [K_i, P_j] = m\,\delta_{ij} X_{(\theta)} \\ [\text{any other}, f \otimes X_{(\theta)}] = (L_{\text{any other}} f) \otimes X_{(\theta)} \end{gathered} \tag{28}$$

Let us consider now the Lie derivative of Θ_W with respect to the vector field $i\zeta f \partial/\partial\zeta$. Instead of being zero, now we find

$$L_{f \otimes X_{(\theta)}} \Theta_W = d i_{f X_{(\theta)}} \Theta_W = df \tag{29}$$

i.e., Θ_W is no longer strictly invariant. To obtain strict invariance under the action of the local $U(1)$ a new modification of the quantum manifold is required: it is necessary to add to the connection form Θ_W a new term $\gamma = \gamma_i dx^i + \gamma_0 dt$ (in fact, a difference of connections) whose components γ_0, γ_i transform under $U(1)(t, x)$ as the space-time gradient of the function f which appears in $f \otimes X_{(\theta)}$. Additional conditions on γ are obtained by requiring strict invariance of $\Theta_W + \gamma$ under the complete group $\tilde{G}_{(m)}(t, x)$.

The algebra of the local $\tilde{G}_{(m)}$ may be realized on the variables $(t, \mathbf{x}, \mathbf{p}, \boldsymbol{\gamma}, \gamma_0, \zeta)$ by[3]

$$
\begin{aligned}
P_{(0)} &= \frac{\partial}{\partial t} \\
P_{(i)} &= \frac{\partial}{\partial x^i} \\
J_{(i)} &= \varepsilon_{ij\cdot}^{\;\;k} x^j \frac{\partial}{\partial x^k} + \varepsilon_{ij\cdot}^{\;\;k} p^j \frac{\partial}{\partial p^k} + \varepsilon_{ij\cdot}^{\;\;k} \gamma^j \frac{\partial}{\partial \gamma^k} \\
K_{(i)} &= t\frac{\partial}{\partial x^i} + m\frac{\partial}{\partial p^i} - i\zeta m x_i \frac{\partial}{\partial \zeta} + \gamma_i \frac{\partial}{\partial \gamma_0} \\
f \otimes X_{(\theta)} &= -\frac{\partial f}{\partial x^i}\frac{\partial}{\partial \gamma_i} + \frac{\partial f}{\partial t}\frac{\partial}{\partial \gamma_0} - i\zeta f \frac{\partial}{\partial \zeta}
\end{aligned}
\tag{30}
$$

With the help of (30), now it is not difficult to show that

$$L_X \Theta'_W = 0 \tag{31}$$

$\forall X$ of (30) where $\Theta'_W \equiv \Theta_W + \gamma$.

As for the finite action of the group on the variables $(\gamma_0, \boldsymbol{\gamma})$ we obtain

$$\boldsymbol{\gamma}' = R\boldsymbol{\gamma}, \qquad \gamma^0 = \gamma^0 + \mathbf{v}\cdot R\boldsymbol{\gamma} \tag{32}$$

under a rotation and a boost and

$$\boldsymbol{\gamma}' = \boldsymbol{\gamma} + \nabla f, \qquad \gamma^0 = \gamma^0 + \frac{\partial f}{\partial t} \tag{33}$$

under an element of $U(1)(t, \mathbf{x})$. It is clear that (32) and (33) correspond to the transformations of the e.m. potentials $\phi \equiv \gamma^0, \mathbf{A} \equiv \boldsymbol{\gamma}$ under Galilean transformations and that, accordingly, the requirement of invariance under $\tilde{G}_{(m)}(t, \mathbf{x})$ has led naturally to the introduction of the interaction with the e.m. field. With the physical notation for $(\gamma^0, \boldsymbol{\gamma})$ the quantization (connec-

[3]For the definition of a local algebra see, e.g., R. Hermann, (1970). *Lie Algebras and Quantum Mechanics*. W. A. Benjamin, New York.

tion) form Θ'_W is written

$$\Theta'_W = (p_i - A_i)\,dx^i - (H - \phi)\,dt + \frac{d\zeta}{i\zeta} \tag{34}$$

Equation (34) clearly shows that the new quantization form is obtained from that of the free case by means of the minimal substitution [in (34) the charge of the particle has been set equal to unity]. In the next section we show how (34) can be used to describe the motion of the particle in a magnetic field.

7. PARTICLE IN AN EXTERNAL MAGNETIC FIELD

If we now consider $\phi, \mathbf{A}$ as given functions on $\mathbb{R}^3 \times \mathbb{R}$ we may follow the steps of Section 4 to describe the motion of a quantum particle in the presence of an external e.m. field. We shall perform the calculation explicitly for the case of a magnetic field described by the external vector potential $\mathbf{A}(\mathbf{x})$. Putting explicitly the charge of the particle (Gaussian units) and the Planck constant, (34) reads

$$\Theta_W = p_i\,dx^i - H\,dt - \frac{q}{c}A_i\,dx^i + \hbar\frac{d\zeta}{i\zeta} \tag{35}$$

Such an expression is globally valid if the magnetic field is "physical," i.e., if $\nabla \cdot \mathbf{B} = 0$ (the presence of a Dirac monopole will be discussed in Section 8). The exterior differential of (35) is given by

$$d\Theta_W = dp_i \wedge dx^i - \frac{p^i}{m}dp_i \wedge dt - \frac{q}{c}\frac{\partial A_i}{\partial x^j}dx^j \wedge dx^i \tag{36}$$

Using (16) for the general expression of the vector field on W, the equations which determine the characteristic vector field are

$$i_X\Theta_W = p_i X^i - HX_t - \frac{q}{c}A_i X^i + \hbar X_\zeta = 0 \tag{37}$$

$$i_X d\Theta_W = X_{i*}\,dx^i - X^i\,dp_i - \frac{p^i}{m}X_{i*}\,dt + \frac{p^i X_t}{m}dp_i - \frac{q}{c}(\partial_j A_i - \partial_i A_j)X^j\,dx^i = 0 \tag{38}$$

from which we get

$$X^i = \frac{p^i}{m} X_t$$

$$X_{i^*} = \frac{q}{c}(\partial_j A_i - \partial_i A_j) X^j$$

$$\frac{p^i}{m} X_{i^*} = 0 \tag{39}$$

$$X_\zeta = \frac{1}{\hbar}\left(H X_t - p_i X^i + \frac{q}{c} A_i X^i \right)$$

$$X_t = X_t$$

Putting again $X_t = 1$ the characteristic vector field defines the following differential system:

$$\frac{dx^i}{dt} = \frac{p^i}{m}, \qquad \frac{dp_i}{dt} = \frac{q}{c} F_{ij} \frac{p^j}{m} \qquad (\Rightarrow |\mathbf{p}| = \text{const})$$

$$\frac{d\zeta}{i\zeta} = \frac{1}{\hbar}\left(-\frac{\mathbf{p}^2}{2m} + \frac{q}{c}\frac{\mathbf{A}\cdot\mathbf{p}}{m} \right) dt \tag{40}$$

The last equation of (40) may be partially integrated with the result

$$\zeta = C \exp\left(-\frac{i}{\hbar}\frac{\mathbf{p}^2}{2m} t \right) \exp\left(\frac{iq}{\hbar c m} \int_\Gamma \mathbf{A}\cdot\mathbf{p}\, dt \right)$$

which may be written

$$\zeta = \zeta_0 \exp\left(\frac{iq}{\hbar c} \int_\Gamma \mathbf{A}\cdot d\mathbf{x} \right) \tag{41}$$

where ζ_0 is the solution in the noninteracting case [cf. (18)] and Γ is a physical trajectory solution of the first two equations of (40), i.e., a classical solution. Note that when $F_{ij} = 0$ ($F_{ij} dx^i \wedge dx^j$ is the symplectic form associated with the magnetic field) in a region of the physical space, we have that $\mathbf{A}\cdot\mathbf{dx}$ is locally of the form $d\chi$ and thus we find

$$\oint_\Gamma \mathbf{A}\,\mathbf{dx} = \oint_\Gamma d\chi = 0 \tag{42}$$

for a closed trajectory contained in the local chart where $\mathbf{A}\cdot d\mathbf{x} \equiv A = d\chi$.

This result exhibits the physical inobservability of the potential **A** which appears in the third equation of (40) when there is no magnetic field ($F_{ij}=0$); the gauge invariance of the result is evident by making an analogous consideration. When the experimental setting does not allow (42) to hold, the potential (or rather, its cohomology class) becomes observable; this is the case of the celebrated Aharonov–Bohm effect.

8. QUANTIZATION OF THE ELECTRIC CHARGE IN PRESENCE OF A DIRAC MONOPOLE.

As a final application of the formalism, we derive now Dirac's quantization condition. As noted in Section 7, the posibility of writing Θ_W for a charged particle in the presence of a magnetic field is associated with the absence of magnetic monopoles ($\operatorname{div}\mathbf{B}=0$), because in this case may write the magnetic field *globally* as the differential of the 1-form $A=\mathbf{A}\,d\mathbf{x}$. In the presence of a Dirac monopole F is no longer exact and the expression for Θ_W given by (35) is valid only locally. Nevertheless it is always possible to write the expression corresponding to its differential which, in the presence of monopoles, we shall denote simply as Ω_W since there is no globally defined 1-form Λ_W such that $\Omega_W=d\Lambda_W$. Ω_W has the expression

$$\Omega_W = dp_i \wedge dx^i - \frac{p^i}{2m} dp_i \wedge dt - \frac{q}{c} F_{ij} dx^i \wedge dx^j \tag{43}$$

The quantization process should now follow the already mentioned steps of transporting Ω_W to the manifold of solutions U by solving the classical problem to construct then the quantum manifold (P,Θ). Nevertheless, we shall bypass these intermediate steps[4] by using the fact that there are no closed classical trajectories for a particle moving in the field of a magnetic monopole. This means that there is no quantization for the particle trajectories, and that accordingly Ω_W may be written on the manifold of solutions as

$$\Omega_W|_U = d\lambda - \frac{q}{c} F_{ij} dx^i \wedge dx^j \tag{44}$$

[4] The question of characterizing the manifold of solutions for the problem of a charged particle in the presence of a monopole is an interesting problem which, as far as we know, has not been solved in the published literature. For the case of the Coulomb problem, see Simms (1973) and references therein.

where the elements of F_{ij} are the components of the magnetic field of the monopole, $\mathbf{B}=g\mathbf{r}/r^3$ and g is its magnetic charge. Now it is clear that $d\lambda$ is an exact form, and so of integer (in fact, zero) class, and that the integrality condition (Kostant, 1970; Śniatycki, 1980; Trautman, 1979) is then only relevant for the (nonexact) part of $\Omega_{W'}(-q/c)F_{ij}dx^i \wedge dx^j$. Thus the quantization condition for the manifold $(U, d\lambda - q/cF_{ij}dx^i \wedge dx^j)$ imposes the condition of being of integer class on the symplectic form associated with the magnetic monopole field. To make this condition explicit we integrate this form on a sphere centered in the monopole, which may be taken as the coordinate origin, and take as charts (a,b) [the bundle is no longer trivial (Yang, 1975, 1977; Wu and Yang, 1977)] the two sterographic projections from the north and south poles. Thus, we have

$$hn=\int \frac{q}{c}F_{ij}dx^i \wedge dx^j=\frac{q}{c}\left(\oint \mathbf{A}_{(a)}d\mathbf{x}-\oint \mathbf{A}_{(b)}d\mathbf{x}\right) \tag{45}$$

where in (a) $0\leqslant\theta\leqslant\pi/2+\varepsilon$ and in (b) $\pi/2-\varepsilon\leqslant\theta\leqslant\pi$; $0\leqslant\varphi<\pi$ in both. With

$$\left(A_{(a)}\right)_\varphi=\frac{g}{r}\frac{(1-\cos\theta)}{\sin\theta} \qquad \left(A_{(b)}\right)_\varphi=-\frac{g}{r}\frac{1+\cos\theta}{\sin\theta} \tag{46}$$

$$\left(A_{(a,b)}\right)_{r,\theta}=0$$

we get at the equator

$$A_a - A_b = d\varphi_{ab} \tag{47}$$

where φ_{ab} is the 1-cocycle determined by $A_{(a)}, A_{(b)}$; $\varphi_{ab}=2g\varphi/r$. The integrals of (45) give the monopole charge inside the sphere times 4π and we obtain

$$nh=\frac{4\pi gq}{c} \tag{48}$$

which is Dirac's quantization condition (Dirac, 1931, 1948)

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